Motional Smearing of Electrically Recovered Couplings Measured from Multipulse Transients

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ABSTRACT: The measurement of residual dipolar and quadrupolar coupling constants in the liquid phase by using an electric field to destroy the isotropic nature of molecular tumbling is complicated by charge-induced turbulent motion. In many cases this motion is due to charge injection at electrode surfaces, an effect that leads to an apparent removal of electrically recovered anisotropic spectral splittings when measured from a spin-echo envelope modulation produced by a train of radio frequency (rf) pulses. To understand this averaging, the effect of quadrupolar couplings and enhanced molecular diffusion on free-induction, spin-echo, and Carr–Purcell signals is analytically determined in the special case of homogeneous rf pulses. Additional signal damping due to rf inhomogeneity and coupling constant heterogeneity is determined by numerically extending the kernel formalism introduced by Herzog and Hahn to understand spin diffusion in solids. Finally, the merit of the numerical approach is tested by comparison with analytical results for homogeneous rf pulses and experimental results for perdeuterated nitrobenzene involving inhomogeneous rf pulses and coupling heterogeneity. © 2001 John Wiley & Sons, Inc. Concepts Magn Reson 13: 171–189, 2001

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Nuclear magnetic resonance (NMR) spectroscopy has moved to the forefront of chemical structure determination techniques in solids, liquids, and gases (1). As is well known, measurement of chemical shifts and scalar couplings as spectral peak positions and splittings yields information about primary and secondary structure. Three-dimensional tertiary and quaternary structural details can be obtained from changes in peak intensity via the nuclear Overhauser effect (nOe) (2). As pioneered by Bothner-by et al. (3), Prestegard et al. (4), and Tjandra and Bax (5), higher resolution molecular structures can be obtained in the liquid phase by measuring residual dipolar couplings. These couplings, manifested as NMR line splittings, display a less severe dependence on internuclear distance than do relaxation-based
nOe measurements. Typically, partially aligned liquid crystalline media such as magnetically anisotropic phospholipid bicelles (6), bacteriorhodopsin purple membranes (7), and bacteriophage (8) are used to destroy the isotropic nature of tumbling solute macromolecules. This collis-

ional ordering ultimately changes the average $\langle 3 \cos^2 \theta - 1 \rangle$ from zero to a nonzero value, where $\theta$ is the angle between the magnetic field $B_0$ and the principal axis of the alignment tensor. For molecules with large magnetic anisotropy $\Delta \chi$, magnetic alignment proceeds without using a li-

quid crystalline medium. The average $\langle 3 \cos^2 \theta - 1 \rangle$ becomes $2 \Delta \chi B_0^2 / 15kT$ where the principle axis is that of the magnetic susceptibility tensor $\chi$, $k$ is Boltzmann’s constant, and $T$ is temperature (3). At room temperature and standard magnetic field strengths the observed dipolar frequency $D$, or equivalently the residual dipolar coupling, changes from $D = 0$ to $D = \langle 3 \cos^2 \theta - 1 \rangle \omega_D / 5kT$ for two $^1$H nuclei separated by 1 Å, $\omega_D = 120$ kHz; therefore the residual coupling lies in the range $1.2$ Hz $\leq D \leq 120$ Hz, an easily measured quantity.

Research in our laboratory has focused on using electric fields to achieve molecular alignment and recover residual dipolar and quadrupolar couplings (9, 10). This approach has the advantage of field strength variability and switching on the time scale of a multidimensional NMR experiment—characteristics not shared by magnetic, collisional, and mechanical molecular alignment. As first shown by Sears and Hahn (11) for small molecules with dipole moment $p$, the average $\langle 3 \cos^2 \theta - 1 \rangle$ reduces to $2pE/kT$ $^2/15 \approx 10^{-4}$ for electrical orientation in tens of kV/cm electric fields $E$. Therefore, residual couplings can be recovered using electric fields in direct analogy to the magnetic orientation introduced above. Application of these recoupling strategies to solutions of complex macromolecules immediately increases spectral congestion and thus requires simplification via multipulse $n$-dimensional NMR techniques. However, as noticed by Hilbers and MacLean (12), electrically induced molecular motion in polar organic solutions leads to sub-

stantial line narrowing in the $^2$H NMR spectrum of aligned perdeuterated nitrobenzene, where residual quadrupolar couplings $Q = \langle 3 \cos^2 \theta - 1 \rangle \omega_Q$ earmark electrically induced anisotropy. More detailed analysis of this turbulent motion in perdeuterated nitrobenzene using $^2$H magnetic resonance imaging afforded an electrically dependent diffusion coefficient of $D_0 = 1.08 \times 10^{-5}$ cm$^2$/s + $(3.33 \times 10^{-3}$ cm$^3$/kV$^2$s)$E^2$ where $E$ is in units of kV/cm (9). This enhanced molecular motion forces conventional two-dimensional methods of correlating couplings with isotropic spectra to fail. In these circumstances one would naturally expect that the Carr–Purcell echo train would minimize the effect of enhanced diffusion on the observed transient (13). Unfortunately, it is impossible to ensure adequate radio frequency (rf) homogeneity across an NMR sample held between two conducting electric field plates. The pulse-to-pulse incoherent change in tip angle to which a diffusing spin is subjected therefore results in a complicated smearing of residual coupling frequencies that typically modulate multipulse spin echoes. Furthermore, the consequences of exchange or diffusion within a distribution of coupling constants, conditions produced by electrode end effects and electrical double layers, are difficult if not impossible to predict a priori.

The effect of diffusion on NMR transient signals has been exhaustively explored since the discovery of the spin echo (14, 15). Early models of motional line narrowing in liquids were based on descriptions of the effects of exchange narrowing in paramagnetic resonance (16) and of spin diffusion in NMR spectra of solid samples (17). In the special case of a Gaussian joint probability distribution of positions, as, for example, in the case of unrestricted Brownian motion, the cumbersome computational techniques originally used can be replaced with an approach using an exponential spatial correlation function (18). Indeed, analytic expressions have been developed for the spin-echo amplitude in the presence of motion following any number of homogeneous rf pulses (19). Additionally, it has been shown that in certain limits, the exponential correlation function approach can be extended to describe situations involving bounded diffusion (20), a process necessarily requiring non-Gaussian probability distributions. Although the transient multipulse NMR signal was calculated exactly following any number of inhomogeneous rf pulses in the absence of motion (21), a systematic treatment of the effects of rf inhomogeneity in the presence of diffusion is more complicated. This is because the correlation function approach breaks down when more than one rf pulse of tip angle other than $\pi$ is used. The effect of diffusion within a distribution of couplings that is different from the distribution of the inhomogeneous field typically used also cannot be described using the correlation function approach. For these more severe and
experimentally realistic conditions, an approach pioneered by Herzog and Hahn (17) and later refined by Klauder and Anderson (22) for the three $\pi/2$ pulse-stimulated echo in a solid system undergoing spin diffusion can be used. The recent interest in studying $I = 1/2$ systems participating in spin diffusion in the solid state (23) and in liquids undergoing translational motion under both controllable (24) and extreme conditions (25), combined with the added complexity inherent in our experiments involving electrically aligned $I = 1$ nuclei displaying residual quadrupolar couplings, prompted an investigation of the effects of rf inhomogeneity and diffusion within coupling distributions on multipulse NMR transients.

The first three sections of this paper are intended as a review of the calculation of free-induction, spin-echo, and multiple-echo signals using homogeneous rf pulses in the presence of motion. The generation of the analytic expression for these signals for one isochromat in an $I = 1$ system with residual quadrupolar coupling $Q$ using a Liouville space formalism is explained. Before averaging these isochromat signals, a brief explanation of the origins of the exponential correlation function approach to treating motion as introduced by Uhlenbeck, Ornstein, Wang, Chandrasekhar, and Kac (26–30) is included. In the third section the exponential correlation function appropriate for a Gaussian joint probability distribution is used to calculate observable NMR transients. Finally, the effects of rf pulses having arbitrary tip angles, rf inhomogeneity, and coupling heterogeneity are included by numerically extending the kernel formalism introduced by Herzog and Hahn (17).

**GENERATING THE SIGNAL**

A reasonable starting point for the calculation of the signals resulting from application of one rf pulse, the spin-echo experiment, and the Carr–Purcell sequence to a system of $I = 3/2$ nuclei is the solution to the Liouville von Neumann equation. To account for molecular motion in both inhomogeneous static and rf magnetic fields, these solutions are then averaged over appropriate Fokker–Planck probability functions for diffusion (31). A few initial approximations can be made: (1) an rf pulse can be considered a perfect zero-time rotation of $\theta$ about the $x$ or $y$ axes in the rotating frame, (2) during the rf pulse all other effects can be neglected, and (3) between rf pulses both the frequency offset and residual quadrupolar coupling are operative. The straightforward functional form of the evolution Hamiltonians combined with the repeated $\tau$–rf pulse–$\tau$ building block of both the spin-echo and Carr–Purcell sequences suggest that the solution $|\rho(t)\rangle = R(t)|\rho(0)\rangle$ for the Liouville von Neumann equation in the Liouville space for an $I = 1$ nucleus spanned by the nine basis states

$$
|I_x\rangle \quad |I_y\rangle \quad |I_z\rangle
$$

$$
|I_x^2 - I_y^2\rangle = |dqr\rangle
$$

$$
|I_x I_y - I_y I_x\rangle = |dqi\rangle
$$

is a physical way to proceed. The abbreviations $|a_x\rangle$, $|a_y\rangle$, $|dqr\rangle$, and $|dqi\rangle$ are used to represent the antiphase $x$ and $y$ magnetization and the real and imaginary double quantum states, respectively. The state vector $|\rho(0)\rangle$ is the initial condition of the system, here taken to be in the high temperature limit along the $z$ direction as $(h \omega_0 / 3kT) |I_z\rangle$ where $h$ is Planck’s constant and $\omega_0$ is the Larmor frequency. As written here, $R(t)$ is a time-dependent rotation matrix that projects $|\rho(0)\rangle$ into the state $|\rho(t)\rangle$ at the time $t$ and is composed of time sequenced products of rotations due to the evolution Hamiltonians and the rf pulses. The matrix form of the rotation operators for the offset Hamiltonian $R_o(t)$, the residual quadrupolar coupling $R_Q(t)$, and the rf pulse Hamiltonian $R_x(t)$ in the basis of the nine states shown in Eq. [1] is provided in the Appendix. Since the frequency offset and residual quadrupolar coupling Hamiltonians commute, the order of the application of their respective rotation operators as $R_x(t)R_Q(t)$ or $R_Q(t)R_x(t)$ is irrelevant. For the $\tau$–rf pulse–$\tau$ building block mentioned above, $R(2\tau) = R_Q(2\tau)R_x(\tau)R_Q(\tau)R_x(\tau)$, whereas for free evolution following one rf pulse of tip angle $\theta$ $R(t) = R_Q(t)R_x(t)$. In general, both $R_x(t)$ and $R_Q(m\tau)$ depend upon the time of their application $t = m\tau$ due to motion in the inhomogeneous field. Therefore, the size of the matrix elements in $R(m\tau)$ will change from pulse to pulse. By labeling the windows of the Carr–Purcell train consecutively, the generic time-dependent building block for each $\theta$ pulse $m$ becomes

$$
R_{m}((m + 1)\tau) = R_{Q}((m + 1)\tau)R_{x}((m + 1)\tau)
$$

$$
\cdot R_{x}(m\tau)R_{Q}(m\tau)R_{x}(m\tau).
$$

[1]
The overall state vector following \( n \) rf pulses is therefore
\[
|\rho[t = (n + 1)\tau]\rangle = \prod_{m=1}^{n} R_m((m + 1)\tau)R_y(0)|\rho[0]\rangle \tag{2}
\]
which can simply be related to an observable signal \( \tilde{I}_z \) as \( \langle I_z | \rho[t]\rangle \), or the amplitude of the \( |I_z\rangle \) component of \( |\rho[t]\rangle \),
\[
\tilde{I}_z \propto \langle I_z | \prod_{m=1}^{n} R_m((m + 1)\tau)R_y(0)|I_z\rangle \tag{3}
\]
\[
\text{since } |\rho[0]\rangle \propto |I_z\rangle.
\]

A STATEMENT ON AVERAGING

Before moving on to averaging the signals generated by the above machinery for different pulse sequences, consider the basic components of the signal in the limit of homogeneous rf pulses and a single residual quadrupolar coupling constant \( Q \). The signal can always be expressed in terms of sines and cosines of sums of phases \( \phi(t) \) that are proportional to integrals over the evolution frequencies \( \omega(t) \) when the pulses are off. Here \( \omega(t) \) is a random function of time due to free precession in a magnetic field gradient \( G \) as \( \omega(t) = G \cdot \mathbf{r}(t) \) where \( \mathbf{r}(t) \) is the incoherent variation of the displacement due to molecular diffusion. The distribution of \( r(t) \) or equivalently \( \omega(t) \), \( P[\omega(t_1), \omega(t_2)] \) relating \( \omega(t) \) to itself at different times \( t_1 \) and \( t_2 \) is taken to be Gaussian at all times, an assumption motivated by the observed liquids NMR line shape. Although the sample is cylindrically symmetric, application of magnetic field shims commonly yields either Lorentzian- or Gaussian-shaped peaks. This spectrum can be accounted for by assuming a linear magnetic field gradient along the \( z \) direction with a change in the sample distribution function from uniform, as would be the case for a cylindrically symmetric sample, to Gaussian. Therefore, this model for molecular motion assumes an overall Gaussian spectrum with the consequences of diffusion being the random incoherent exchange of isochromats as a function of time. Since the distribution function for \( \omega(t) \) is Gaussian, the distribution for the phase \( \phi(t) \) which is the integral of \( \omega(t) \) from \( t_1 = 0 \) to \( t_1 = t \) will also be a Gaussian function given here by
\[
P[\phi(t)] = \frac{1}{\sqrt{2\pi\langle \phi^2(t) \rangle}} e^{\frac{-\phi^2(t)}{2\langle \phi^2(t) \rangle}} \tag{4}
\]
with width \( \langle \phi^2(t) \rangle \). The value of the average of the mean square width
\[
\langle \phi^2(t) \rangle = \int_0^t \int_0^t \omega(t_1)\omega(t_2) \, dt_1 \, dt_2 \tag{5}
\]
can be determined by realizing that the average implied by the brackets on the right-hand side of Eq. [5] is represented by the integral over the Fokker–Planck conditional probability for \( \omega(t) \) due to diffusion
\[
P[\omega(t), \omega(0)] = \frac{1}{\sqrt{2\pi\omega_0^2(1 - e^{-2\Gamma t})}} \cdot \exp \left( -\frac{(\omega(t) - \omega(0)e^{-\Gamma t})^2}{2\omega_0^2(1 - e^{-2\Gamma t})} \right) \tag{6}
\]
where \( \Gamma \) is an exchange rate. In more complicated cases the function \( P[\omega(t), \omega(0)] \) can be taken to more truly reflect the actual boundary conditions in the problem. However, as we shall see, the Gaussian form for \( P[\omega(t), \omega(0)] \) does predict experimental observations such as motional line narrowing due to free precession during diffusion in inhomogeneous static magnetic fields. It should be clear that at the time \( t = 0 \) the distribution in Eq. [6] is Gaussian with width \( \omega_0 \). Equation [6] can be used to determine the value of the correlation function of the frequency in Eq. [5] as
\[
\langle \omega(t_1)\omega(t_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega(t_1)\omega(t_2) P[\omega(t_1)] \cdot P[\omega(t_2), \omega(t_1)] \, d\omega(t_1) \, d\omega(t_2) = \omega_0^2 e^{-\Gamma|t_1 - t_2|} \tag{7}
\]
which in turn be used to determine the mean square width \( \langle \phi^2(t) \rangle \) in Eq. [5] as
\[
\langle \phi^2(t) \rangle = \frac{2\omega_0^2}{\Gamma^2}(e^{-\Gamma t} - 1 + \Gamma t) \tag{8}
\]
It is this width that completely determines the decay behavior of the observed transient signal following a spatially homogeneous rf pulse—a pulse that behaves like a zero-time rotation of the density operator by a given tip angle (a “crisp” pulse). Consideration of spin echo or multiple echo signals with homogeneous rf pulses requires
an additional integral over the correlation function \( \langle \omega(t_1) \omega(t_2) \rangle \) in the spirit of Eq. [7]. To avoid later confusion and to introduce some notation this integral is now defined. The two-dimensional integral over the correlation function at various times or windows of time \( t \) in a multiple-pulse sequence is calculated from Eq. [7] for \( n > 0 \) to be

\[
\Phi_n = \int_0^t \int_0^{(n+1)t} \omega(t_1) \omega(t_2) \, dt_1 \, dt_2
\]

\[
= \frac{\omega_n^2}{\Gamma^2} \left( e^{-(n+1)\Gamma t} + e^{-(n-1)\Gamma t} - 2e^{-n\Gamma t} \right) \quad [9]
\]

while for \( n = 0 \), \( \Phi_0 = \langle \phi^2(t) \rangle /2 \) where \( \langle \phi^2(t) \rangle \) is given in Eq. [8]. The correlation function in Eq. [7] and its integral \( \Phi_n \) in Eq. [9] can be used to define the functional form for the NMR signal corresponding to application of one rf pulse and any subsequent number of \( \pi \) rf pulses as long as they are homogeneous. In the case of one inhomogeneous rf pulse, the signal can also be analytically determined for special rf field spatial distributions using Eqs. [8] and [9]. However, in the more general cases of a series of rf pulses having tip angles deviating from \( \pi \), or for inhomogeneous rf pulses, or for coupling heterogeneity an iterative or numerical approach will be adopted.

**ANALYTICAL EXPRESSIONS FOR THE SIGNAL**

The observed signal in the presence of molecular diffusion and static magnetic field inhomogeneity can now be reviewed. The NMR free induction signal \( F(t) \) due to a residual quadrupolar coupling \( Q \) in an \( I = 1 \) system and molecular diffusion following a crisp \( \pi/2 \) rf pulse is

\[
F(t) = \langle I_x \rangle = \left\langle \left| I_x \right| R_0(t) R_y(t) R_y(0) \right| I_x \rangle
\]

\[
= \cos(\pi Q t) \left\langle \left| \int_0^t \omega(t_1) \, dt_1 \right| \right\rangle
\]

\[
= \cos(\pi Q t) \langle \cos(\phi(t)) \rangle
\]

\[
= \cos(\pi Q t) \int_{-\infty}^{+\infty} P(\phi(t)) \cos(\phi(t)) \, d\phi(t)
\]

[10]

where again it is clear that the time-dependent evolution frequency \( \omega(t) = \gamma G \cdot r(t) \) is a random function of time set by the incoherent variation of the displacement \( r(t) \) due to molecular diffusion and the magnetic field distribution in the sample set by the magnetic field gradient \( G \). The accumulated phase \( \phi(t) \) at any time in Eq. [10] is just the integral of \( \omega(t) \) from \( t_1 = 0 \) to \( t_1 = t \). The right hand side of Eq. [10] is the Fourier transform of the Gaussian probability distribution \( P(\phi(t)) \) in Eq. [4] that reduces to another Gaussian

\[
F(t) = \cos(\pi Q t) e^{(\phi^2(t))/2}
\]

\[
= \cos(\pi Q t) e^{-\omega_n^2 / \Gamma^2 \left( e^{-t\Gamma} - 1 + \Gamma t \right)}
\]

[11]

where the relation in Eq. [8] has been introduced \((18)\). This equation predicts exactly what one expects in the various limits of \( \Gamma \). In the case of very slow motion, \( \Gamma \approx 0 \) and Eq. [11] predicts a spectrum of two Gaussian peaks of width \( \omega_n \) split by \( 2Q \), while in the fast motion limit two narrower Lorentzian peaks split by \( 2Q \) with widths \( \omega_n^2 / \Gamma \) are anticipated.

**Spin Echoes**

Now we continue to analyze the case for the \( \pi/2 - \tau - \pi \) spin echo. Here a crisp \( \pi \) pulse is applied at the time \( t = \tau \) that rotates the \( |I_x\rangle \) component of the density operator \( \rho(\tau) \) to the \(-z\) direction and the \( |I_y\rangle \) component to the \(-y\) direction for an \( x\)-phase pulse. For a \( y\)-phase pulse, the \( |I_x\rangle \) component is transformed into the \(-x\) direction and the \( |I_y\rangle \) component is left unaffected. Regardless of pulse phase, the net effect of the \( \pi \) rf pulse on the phase \( \phi(t) \) is to change it from positive before the pulse to negative after the pulse. Therefore, the spin-echo signal is given by

\[
E(2\tau) = \langle \tilde{I}_x \rangle = \left\langle \left| I_x \right| R_0(2\tau) R_y(0) |I_x\rangle \right| \rangle
\]

\[
= \cos(2\pi Q \tau)
\]

\[
\cdot \left\langle \cos \left( \int_0^\tau \omega(t_1) \, dt_1 - \int_\tau^{2\tau} \omega(t_1) \, dt_1 \right) \right\rangle
\]

[12]

which can be used along with the definition of the average in Eq. [7] and the double integrals in Eq. [9] to determine the average mean square phase as

\[
\langle \phi^2(2\tau) \rangle = \left\langle \left( \int_0^\tau \omega(t_1) \, dt_1 \right)^2 \right\rangle
\]

\[
= 4\Phi_0 - 2\Phi_1
\]

\[
= \frac{2\omega_n^2}{\Gamma^2} \left( 4e^{-\Gamma \tau} - e^{-2\Gamma \tau} + 2\Gamma \tau - 3 \right)
\]

[13]
This phase along with the result of the Fourier transform of Eq. [10] shown in Eq. [11] can be used to calculate the signal at \( t = 2\tau \). Note that the \( \tilde{a}_i \) and \( \tilde{a}_y \) antiphase contributions to the density operator are not observable in the signal and that the value of the \( \langle \hat{T}_z \rangle \) component averages to zero for the symmetric \( P_t(t) \) used here. In the case of slow motion the standard \( \Gamma^3 \) decay of the spin-echo signal \( E(2\tau) \) is observed

\[
E(2\tau) = \cos(2\pi Q\tau)e^{-\langle \phi^2(2\tau) \rangle/2} = \cos(2\pi Q\tau)e^{-(2\omega_0^2\Gamma/3)p^3} \tag{14}
\]

whereas for fast motion an exponential decay is predicted

\[
E(2\tau) = \cos(2\pi Q\tau)e^{-\langle \phi^2(2\tau) \rangle/2} = \cos(2\pi Q\tau)e^{-(2\omega_0^2/\Gamma)p^3} \tag{15}
\]

Equation [15] indicates that in the fast motion limit, as diffusion increases, the decay rate \( 2\omega_0^2/\Gamma \) decreases, which leads to an increase in the echo height as a function of \( \tau \).

**Multiple Spin Echoes**

The signal resulting from the application of the Carr–Purcell pulse sequence having \( p \) \( \pi \) pulses phase-alternated between 0 and 180° is calculated by extending Eq. [12] to include multiple refocusing events. Here the echo envelope signal at the time \( t = 2p\tau \) is given by

\[
T(2p\tau) = \langle \hat{T}_z \rangle \approx \langle I_z \rangle = \left\langle \prod_{m=1}^{p} R_m((m + 1)\tau)R_y(0)I_z \right\rangle
\]

\[
= \cos(2\pi Q\tau) \left\langle \cos\left( \int_0^\tau \omega(t_1) \, dt_1 \right) - \int_\tau^{2\tau} \omega(t_1) \, dt_1 + \int_{2\tau}^{3\tau} \omega(t_1) \, dt_1 \right. \\
- \int_{3\tau}^{4\tau} \omega(t_1) \, dt_1 + \cdots \\
- \int_{2(p-1)\tau}^{2p\tau} \omega(t_1) \, dt_1 \right\rangle \tag{16}
\]

Again the critical parameter is the average of the mean square after the \( p \)th \( \pi \) pulse at the time \( t = 2p\tau \) given by

\[
\langle \phi^2(2p\tau) \rangle = \left\langle \left( \int_0^\tau \omega(t_1) \, dt_1 - \int_\tau^{2\tau} \omega(t_1) \, dt_1 + \int_{2\tau}^{3\tau} \omega(t_1) \, dt_1 + \cdots \right) \right. \\
- \int_{2(p-1)\tau}^{2p\tau} \omega(t_1) \, dt_1 \right\rangle ^2 \tag{17}
\]

where the two-dimensional integrals over the correlation function are defined in Eq. [9]. This abbreviated notation straightforwardly allows determination of the echo envelope signal \( T(2p\tau) \) accumulated during the Carr–Purcell pulse train, i.e., the transient observed by measuring the signal at each echo maximum. This signal is obtained from Eqs. [11] and [17] as

\[
T(2p\tau) = \cos(2\pi pQ\tau)e^{-\langle \phi^2(2p\tau) \rangle/2} = \cos(2\pi pQ\tau) \prod_{n=0}^{2p-1} e^{-(2\omega_0^2/\Gamma)(2p-n)\Phi_n} \tag{18}
\]

It is useful to explore the echo envelope signal \( T(2p\tau) \) in both the fast and slow motion regimes. For fast motion \( \Gamma \) goes to infinity, forcing the \( \Phi_n \) in Eq. [14] to zero and leaving only \( \Phi_0 = \omega_0^2\tau/\Gamma \) in Eq. [16] and the Lorentzian envelope signal

\[
T(2p\tau) = \cos(2\pi pQ\tau)e^{-(\omega_0^2/\Gamma)(2p\tau)} \tag{19}
\]

For slow motion \( \Gamma \) goes to zero, allowing expansion of the exponentials in the \( \Phi_n \) functions. Collapsing the series in Eq. [17] in this limit gives the expression

\[
T(2p\tau) = \cos(2\pi pQ\tau)e^{-(\omega_0^2/\Gamma^3)(2p\tau)} \tag{20}
\]

for the envelope signal, indicating that slow motional effects can be minimized by choosing small values for the delay time \( \tau \).
Arbitrary Tip Angle rf Pules

In the limit that $Q = 0$, the above expressions for the free induction, spin-echo, and Carr–Purcell signals yield the same results expected on the basis of the Bloch equations corrected for the effects of diffusion. This standard formalism unfortunately breaks down when either multiple rf pulses having tip angles deviating from $\pi$ or single and multiple inhomogeneous rf pulses are used. In only certain special cases can analytical expressions be derived to describe NMR transients observed during diffusion. For example, the above formalism can be used to describe the effect of triangular inhomogeneous rf field distributions on the free induction signal. Here the tip angle of the rf pulse $\theta$ is related to the absolute value of the displacement $z(t)$ at the time $t$ of the pulse inside an rf coil of length $l$ as $\theta = \theta_0 - \theta_0/2$ at each end. Since $z(t)$ is proportional to the observed resonance frequency as mentioned above via the relation $z(t) = \omega(t)/\gamma G$, the tip angle can be recast as $\theta = \theta_0 - \theta_0/2$. For an inhomogeneous $\theta_0 = \pi/2$ pulse, the initial transverse magnetization at the time $t = 0$ is changed from the unit value above to $\cos[\theta_0 \omega(t = 0)/\gamma G]$, while the remaining magnetization along the $z$ direction changes from zero to $\sin[\theta_0 \omega(t = 0)/\gamma G]$. Consider first the transverse magnetization, where all that is necessary to calculate the signal is the distribution of $\omega(0)$ since cosine is an even function and the absolute value can be dropped. Indeed, this realization followed by propagation under both $R_z(t)$ and $R_y(t)$ coupled with knowledge of the correlation function $\langle \omega(t)\omega(0) \rangle$ given in Eq. [7] will yield an analytical expression for the signal. This expression is not provided here because it does not adequately describe actual experimental conditions, i.e., the spatial dependence of the magnetic field within solenoid and Helmholtz rf coils. Furthermore, even in this simple case one cannot easily deduce a functional form for the average $z$ magnetization, a component that is important for multiple inhomogeneous rf pulses. To perform this average the distribution of the absolute value of $\omega(0)$ is needed because sine is an odd function. In general, an rf pulse transforms a distribution of isochromats $P[\omega(t)]$ at time $t$ along some direction to another direction. In the simple case when $Q = 0$ and the Bloch equations can be used to describe magnetization dynamics, the distribution remaining along the original axis following an inhomogeneous rf pulse is $\cos[\theta_0(\omega(t))]P[\omega(t)]$ while the distribution created along the new orthogonal axis is $\sin[\theta_0(\omega(t))]P[\omega(t)]$. The reason why the standard formalism breaks down should now be clear. Following an inhomogeneous rf pulse reflecting any given rf field profile denoted here by $\theta(z(t)) \propto \theta_0(\omega(t))$, the initial Gaussian distribution $P[\omega(t)]$ before the rf pulse becomes characteristically non-Gaussian following the rf pulse. Consequently, the simple exponential correlation function in Eq. [7], which is at the heart of the standard formalism, does not apply. An approximation to calculation of the correlation function could be attempted; however, such a result depends strongly on the functional form of $\theta_0(\omega(t))$. Additionally, such an analytical result would most likely not be easily extended to $Q \neq 0$ magnetization dynamics where, as shown in the Appendix, more complex tip angle dependences are introduced to the state vector $|\rho(t)|$ during an rf pulse.

A clue to calculating NMR transients in the presence of inhomogeneous rf pulses and diffusion with $Q \neq 0$ comes from numerical extension of an analytical formalism used by Klauder and Anderson (22) but first identified and clearly explained by Herzog and Hahn (17) in connection with spin diffusion in solids. This analytical approach was developed for $I = 1/2$ spin system dynamics and successfully applied to describe stimulated echo formation during diffusion and equally well applies to non-Gaussian probability distributions. The basic calculation involves first determining the spectrum of the state vector $|\rho(\omega(n \Delta t), n \Delta t)|$ at the time $n \Delta t$ from the spectrum of the state vector at an earlier time $(n - 1) \Delta t$. By considering infinitesimally small time steps $\Delta t$, the phase $\phi(t)$ in the rotation operator $R_z(t)$ shown in the Appendix can be reduced to $\omega((n - 1) \Delta t) \omega(n \Delta t) \Delta t/2$, which can be easily approximated as $\omega((n - 1) \Delta t) \Delta t$. Similarly the phase $\phi(t)$ in the rotation operator $R_y(t)$ shown in the Appendix becomes $Q \Delta t$. The product of rotation operators $R_y(t)R_z(t)$ for small $\Delta t$ indicates that the spectrum of the state vector at the time $(n - 1) \Delta t$ will contribute a new moment to the spectrum of the state vector at the time $n \Delta t$. The result of the phase fluctuations prescribed by the rotation operator $R_y(\Delta t)$ $R_z(\Delta t)$, which indicates that nuclei migrate out of initial frequencies $\omega((n - 1) \Delta t)$ to final specific values.
frequencies \( \omega(n \Delta t) \) can be expressed by the integral

\[
|p[\omega(n \Delta t), n \Delta t] = \int_{-\infty}^{\infty} P[\omega(n \Delta t), \omega((n - 1) \Delta t)] \\
\cdot R_{Q}(\Delta t)R_{Q}(\Delta t)|p[\omega((n - 1) \Delta t), \\
(n - 1) \Delta t] d\omega((n - 1) \Delta t) \quad [21]
\]

where the conditional probability is given in Eq. [6] by replacing \( t \) with \( \Delta t \). The signal \( \langle \tilde{I}_{x}(n \Delta t) \rangle \) at each time step \( n \Delta t \) can be determined by integrating the product of \( \langle I_{x} \rangle \) with Eq. [21] over \( \omega(n \Delta t) \) as

\[
\langle \tilde{I}_{x}(n \Delta t) \rangle = \langle \tilde{I}_{x}|p[\omega(n \Delta t), n \Delta t] \rangle \\
= \int_{-\infty}^{\infty} \langle \tilde{I}_{x}|p[\omega(n \Delta t), n \Delta t] \rangle d\omega(n \Delta t) \quad [22]
\]

The effects of inhomogeneous rf pulses can be included at both the time \( t = 0 \) in the distribution of \( |p[\omega(0), t = 0] \rangle \) and at other times by replacing \( |p[\omega((n - 1) \Delta t), (n - 1) \Delta t] \rangle \) with \( R_{Q}(n \Delta t) |p[\omega((n - 1) \Delta t), (n - 1) \Delta t] \rangle \). The rotation operator \( R_{Q}(n \Delta t) \) as shown in the Appendix depends on position via the spatial dependence on tip angle \( \theta(\omega(t)) \) at the time \( t \).

### DISTRIBUTION OF COUPLINGS

The situation becomes even more complex when a distribution of residual quadrupolar couplings \( P[Q] \) is assumed and exchange within this distribution is allowed. In principle the isochromatic signal is a function of \( \phi(t) \) and the integral over \( Q(t) \), therefore, integration of this isochromatic signal over the product of distributions

\[
P \left[ \int Q(t') dt' \right] P[\phi(t)] d \left[ \int Q(t') dt' \right] d\phi(t)
\]

will yield the appropriate motionally averaged result. The challenge is then to determine the appropriate distribution \( P[|Q(t') dt'] \) for the signal phase \( |Q(t') dt' \) due to the residual quadrupolar coupling. Although it is safe to choose \( P[\omega(t)] \) and ultimately \( P[\phi(t)] \) as Gaussian distributions since the magnetic field shims are adjusted to force our cylindrically symmetric sample to display Gaussian or Lorentzian peaks in the absence of motion, it is not appropriate to use this highly symmetric function for the distribution of \( Q \) or for \( |Q(t') dt' \) when there is motion. To understand how to proceed, consider the actual experimental situation. Two 1-cm diameter plates separated by 4 mm are charged to develop \( E = 10\text{--}100 \text{ kV/cm} \) electric fields across a cylindrical sample volume. These values for \( E \) correspond to the applied electric field, i.e., the voltage applied to the plates divided by their separation. Optical experiments in polar organic solution indicate that for this type of electric field cell, the electric field is not constant throughout the sample volume but instead varies roughly as the square of the displacement from the \( z = 0 \) center of the sample (32). Therefore, the minimum electric field occurs in the center of the cell with maxima at the face of each electrode. The quadratic dependence of \( E \) on displacement indicates that the residual coupling \( Q \), which depends on the square of the electric field, will also have a quadratic functional form. Assuming that \( E = E_{0} + \bar{E}z(t)^{2} \) between the electrode plates, the residual coupling \( Q \) up to terms quadratic in \( z(t) \) can be rewritten as \( Q = Q_{0} + \bar{Q}z(t)^{2} \) where \( Q_{0} = 2(pE_{0}/kT)^{2} \omega_{0}/15 \) and \( \bar{Q} = 4(p/kT)^{2}E_{0}\bar{E} \omega_{0}/15 \). In a perfectly homogeneous magnetic field, the effect of this quadratic dependence of \( Q \) on \( z(t) \) for a cylindrically symmetric sample in the absence of motion is to change the spectrum from two delta function-like lines centered at \( \pm Q_{0} \) to two broad lines ranging in frequency from \( \pm Q_{0} \) to \( \pm (Q_{0} + \bar{Q}z(t)^{2}/4) \), respectively, for nonzero \( \bar{Q} \) values. For \( \bar{Q} \neq 0 \) and no motion, the maximum spectral intensity occurs at \( \pm Q_{0} \) and decays toward lower absolute frequency, changing the distribution from flat and uniform to peaked will have little noticeable effect on the observed spectrum as long as the peak width is large. For narrower Gaussian or Lorentzian distributions like those used above, the overall spectral features can be recovered by suitable adjustment of both \( Q_{0} \) and \( \bar{Q} \). The addition of molecular motion for \( \bar{Q} \neq 0 \) will also move the maxima of the spectrum. For a uniform distribution, the \( \pm Q_{0} \) maxima will shift toward higher absolute frequency, reflecting the contribution of nonzero mean square displacement to the overall frequency as roughly \( Q_{0} + \bar{Q}z(t)^{2} \). It should now be clear that for an analytical
treatment of the effects of motion on heterogeneous electrically recovered quadrupolar couplings that the distribution $P[\hat{Q}(t') dt']$ is equivalent to $P[Q(t') dz(t') dt']$ much like $P[\phi(t')]$ is equivalent to the $P[\omega(t') dt'] = P[\gamma G|z(t') dt']$ distribution used above. In other words, the distributions for $Q(t)$ and $\omega(t)$ are related by the displacement $z(t)$, and the variables to be integrated to construct averages are $z(t') dt'$ and $z(t')^2 dt'$. Even with this realization, an analytical approach is still intractable. However, the numerical method outlined in Eqs. [21] and [22] is straightforward to implement because the conditional probability for $\omega(t)$ or equivalently $z(t)$ is used. One basically connects $Q(t)$ to $\omega(t)$ with the relation $\omega(t) = \gamma G z(t)$, giving $Q(t) = Q_0 + \bar{Q}(\omega(t)/\gamma G)^2$. Consideration of small time steps $\Delta t$ recasts the phase $[Q(t') dt']$ in the rotation operator $R_{Q}^{\Delta t}$ as $Q_0 \Delta t + \bar{Q}(\omega^2((n-1) \Delta t) + \omega^2(n \Delta t))/2(\gamma G)^2$, which much like the case discussed above for the phase $\phi(t)$ can be reduced to $Q_0 \Delta t + \bar{Q}(\omega^2(n \Delta t))/G^2$. Even though the conditional probability $P[\omega(t), \omega(0)]$ is Gaussian at all times while the distribution for $[Q(t') dt']$ is in reality uniform, errors introduced by using this Gaussian form for $P[Q(t), Q(0)]$ can be compensated for by slight modifications to the parameters $Q_0$ and $\bar{Q}$.

NUMERICAL VALIDITY

Before proceeding with the description of the effects of rf inhomogeneity and coupling heterogeneity on multipulse transients using the numerical formalism outlined in the previous sections, the approach must be tested to ensure consistency with the analytical technique embodied in Eqs. [11], [12], and [18]. The dashed lines in Figs. 1–3 correspond to analytical results derived from Eqs. [11], [12], and [18], while the solid lines represent numerical results computed using Eqs. [21] and [22]. In all figures the residual quadrupolar coupling is taken to be crisp and fixed at $Q = 5 \text{ Hz}$ and the linewidth $\omega_a$ is set to 6 rad/s, a direct reflection of residual splittings and line-widths observed in one-dimensional NMR spectra for perdeuterated samples in our laboratory (10). The transients in Fig. 1 correspond to the free induction signal $F(t)$ following a homogeneous $\pi/2$ rf pulse as the exchange rate $\Gamma$ is increased in the order 0.06, 6, and 60 s$^{-1}$ for Fig. 1(A), (B), and (C), respectively. As expected the increased diffusion does not change the evolution frequency $Q$ but does cause motional narrowing of the observed line manifested as longer lifetimes in the series (A)–(C). Additionally, the agreement between the analytical results from Eq. [11] and the numerical solutions is satisfactory, although there is some deviation in the fast-motion limit. Variation of the number of time steps and dwell time $\Delta t$ suggests that this discrepancy is a normalization error inherent in the numerical approach.

Figure 1 Comparison of analytical (dashed line) and numerical (solid line) results for the free-induction signal corresponding to a 6 rad/s-wide line following a homogeneous $\pi/2$ rf pulse and with a residual quadrupolar coupling of $Q = 5 \text{ Hz}$. In the zero-motion limit in (A), the exchange rate $\Gamma$ was set to 0.06 s$^{-1}$ while $\Gamma = 6$ and 60 s$^{-1}$ in (B) and (C), respectively.
Figure 2 Comparison of analytical (dashed line) and numerical (solid line) results for the spin-echo amplitude following a $\pi/2-\pi$ homogeneous pulse pair as a function of pulse spacing for the same $Q = 5$ Hz, $I = 1$ quadrupolar system used in Figure 1. Again, the exchange rate $\Gamma = 0.06$, 6, and 60 s$^{-1}$ in (A), (B), and (C), respectively.

The agreement between the analytical and numerical approaches persists when one looks at the spin-echo signal $E(2\tau)$ as a function of the time $2\tau$ as shown in Fig. 2. In effectively the zero-motion limit where the rate $\Gamma = 0.06$ s$^{-1}$ as shown in Fig. 2(A), the second rf pulse refocuses all static magnetic field inhomogeneity and no decay is observed. Increasing the rate $\Gamma$ to 6 s$^{-1}$ in Fig. 2(B) causes the echo signal to damp as the isochromatic signals destructively interfere. Finally in Fig. 2(C) where $\Gamma = 60$ s$^{-1}$, the motion is so fast that the diffusion appears to be uncorrelated on the $2\tau = 1$ s time scale. Again, as in Fig. 1, the most serious discrepancy in Fig. 2 between the numerical approach shown as the solid line and the analytical result from Eq. [12] represented by the dashed line for $E(2\tau)$ occurs in the
fast-motion limit. The most significant smearing of the residual quadrupolar coupling \( Q \) occurs when the rate \( \Gamma \) is on the order of the time scale of the experiment. This limit is shown in Fig. 2(B) for \( \Gamma = 6 \text{ s}^{-1} \), where both the analytical and numerical results coincide.

The effect of the largely exponential decay predicted by Eq. [15] and shown in Fig. 2(B) and (C) can be removed by replacing the one \( \pi \) rf pulse in the spin echo sequence with \( n \pi \) rf pulses separated by \( \tau/n \) as prescribed by the Carr–Purcell experiment. The results of this sequence as a function of \( \pi \) pulse for 100 \( \pi \) pulses separated by 10 ms are shown in Fig. 3. Again the most significant deviation between the analytical and numerical approaches is in the fast-motion limit when the rate \( \Gamma = 60 \text{ s}^{-1} \) as shown in Fig. 3(C). Notice also that the damping in Fig. 2(B) for \( \Gamma = 6 \text{ s}^{-1} \) is completely removed by the Carr–Purcell experiment as shown in Fig. 3(B). Furthermore, for this particular parameter set, the analytical results in Fig. 3 indicate that in all cases the Carr–Purcell experiment refocuses the signal, yielding only a crisp well-defined frequency at the residual quadrupolar coupling \( Q \).

**APPLICATION TO INHOMOGENEOUS RF PULSES**

Having verified that the numerical formalism displayed in Eqs. [21] and [22] adequately reproduces analytical results for free induction, spin echo, and Carr–Purcell signals, it is natural to use the numerical approach to determine the corresponding signals in the presence of inhomogeneous rf pulses. Although any functional form can be chosen for the spatial dependence of the rf pulse, here we limit ourselves to the Gaussian tip angle \( \theta(t) = \theta_0 \exp(-a_0^2) \) where the inverse square width \( a \) is in units of \( \text{s}^{-2}/\text{rad}^2 \) and the tip angle at the center of the rf coil \( \theta_0 \) is the crisp value, here either \( \pi/2 \) or \( \pi \). Regardless of the motional regimes explored and the field inhomogeneities investigated, here \( 0.06 \leq \Gamma \leq 60 \text{ s}^{-1} \) and \( 0 \leq a \leq 0.0018 \text{ s}^{-2}/\text{rad}^2 \), respectively, the free-induction signals were exactly like those shown in Fig. 1. It is important to note that for \( a = 0.0018 \text{ s}^{-2}/\text{rad}^2 \), the field at \( z(t) = \pm 1/2 \) drops to half the value at the center while for \( a = 0.00075 \text{ s}^{-2}/\text{rad}^2 \), the field at the same point drops to three fourths the value at the center. For a \( \theta_0 = \pi \) rf pulse these choices mean that the tip angle will be \( \pi/2 \) and \( 3\pi/4 \) at the edges of the sample, respectively. The effects of inhomogeneous rf pulses on the spin-echo signal \( E(2\pi) \) and the Carr–Purcell signal \( T(2p\tau) \) are shown in Figs. 4 and 5. The solid lines in these plots correspond to the numerical simulations shown in Figs. 2 and 3.

Figure 4 Investigation of the effect of rf pulse inhomogeneity on the amplitude of the spin-echo signal for the same \( Q = 5 \text{ Hz system} \) having \( \omega_n = 6 \text{ rad/s zero-motion line width} \). The solid line corresponds to homogeneous rf pulses while the dashed and dot/dashed lines represent Gaussian tip angle profiles. For the dashed line, the rf field at the edge of the sample is taken to drop to 75% of the value at the center while for the dot/dashed line the edge field is 50% of the center value. As in previous figures, the rate \( \Gamma \) increases in the order \( \Gamma = 0.06, 6, \) and \( 60 \text{ s}^{-1} \) in (A), (B), and (C), respectively.
Since only two rf pulses are used in the spin-echo experiment, there are minimal effects of rf inhomogeneity on the echo transient $E(2\tau)$ as shown for $\Gamma = 0.06, 6, 60$ s$^{-1}$ in Fig. 4(A), (B), and (C), respectively. However, introduction of multiple inhomogeneous rf pulses as prescribed by the Carr–Purcell experiment causes the signal $T(2p\tau)$ as a function of pulse $p$ to display pedestal, damping, and frequency shift effects. Figure 5 demonstrates that these effects increase in severity as motion increases for $\Gamma = 0.06, 6, 60$ s$^{-1}$ in (A), (B), and (C), respectively, where the same 100 $\pi$ pulse experiment as in Fig. 3 with a pulse spacing of 10 ms is used. The pedestal and shift of the transients toward higher frequency can be understood by considering all of the components of $|\rho(t)\rangle$. The $x$-phase inhomogeneous rf pulse shown as the rotation operator $R_x(m\tau)$ in the Appendix converts some fraction of the $|I^z\rangle$ component into $|I_x\rangle$ magnetization because $\theta(\omega(2p\tau)) \neq \pi$. Since the $|I^z\rangle$ component evolves under neither the frequency offset nor the residual quadrupolar coupling Hamiltonians, the amount of the $|I^z\rangle$ component converted into $|I_x\rangle$ magnetization translates into a DC offset or signal pedestal. The frequency shift is due to a similar exchange but involves other components of $|\rho(t)\rangle$ such as the double quantum terms $|dqr\rangle$ and $|dq_i\rangle$. In effect, a larger fraction of these double frequency terms can be converted into observable $|I_x\rangle$ magnetization for faster motion, because a given spin has more frequent access to rf pulses deviating from $\pi$. The apparent damping of signal in Fig. 5(B) and 5(C) is most likely a reflection of the inhomogeneous rf pulses converting the observable $|I_x\rangle$ magnetization into the other basis states in $|\rho(t)\rangle$.

**Application to Coupling Heterogeneity**

The effects of coupling heterogeneity with and without molecular motion can also be easily explained with this numerical formalism. The simulations in Figs. 6 and 7 correspond to the free induction decay $F(t)$ and echo train $T(2p\tau)$ signals using homogeneous rf pulses for the same 6 rad/s$^{-1}$/rad$^2$ lines displaying a $Q_0 = 5$ Hz coupling constant. Again in Figs. 6 and 7 the degree of motion is increased in the order $\Gamma = 0.06, 6, 60$ s$^{-1}$ in (A), (B), and (C), respectively. The solid lines in Figs. 6 and 7 correspond to the numerical simulations in Figs. 1 and 3, respectively.
MOTIONAL SMEARING OF ELECTRICALLY RECOVERED COUPLINGS

Figure 6 Investigation of the effect of coupling heterogeneity on the free induction signal following a homogeneous $\pi/2$ rf pulse. These free induction decays were obtained using the same $I = 1$ quadrupolar system used in Figs. 1–5. The solid line corresponds to $\tilde{Q}/(\gamma G)^2 = 0$ s/rad$^2$ while the dashed and dot/dashed lines represent $\tilde{Q}/(\gamma G)^2 = 0.1$ and 1.0 s/rad$^2$ values, respectively. The exchange rate $\Gamma$ increases in the order $\Gamma = 0, 6,$ and $60$ s$^{-1}$ in (A), (B), and (C), respectively.

dashed lines in Figs. 6 and 7 reflect an increase of the $\tilde{Q}/(\gamma G)^2$ value to 0.1 and 1.0 s/rad$^2$, respectively. The effects of coupling heterogeneity are most pronounced in the free induction signals shown in Fig. 6. In the absence of motion and for intermediate motion in Fig. 6(A) and (B), a small increase in $\tilde{Q}/(\gamma G)^2$ produces a longer-lived signal, whereas for fast motion no major effect is recognized. In the slower motion limits, the average of $z(t)^2$ is competing with and amplifying the averaging of $z(t)$, thus producing essentially nar-
rower lines. In the faster motion limit and for increased coupling heterogeneity as shown by the dot/dashed lines in Fig. 6, the average of $z(t)^2$ is substantially different from zero and shorter lifetime frequency shifted transients are observed. All of these effects are truncated by operation in multipulse mode as shown in Fig. 7. For only large $Q/(\gamma G)^2$ values is the damping due to coupling heterogeneity noticeable in the Carr–Purcell signal. More importantly, the Carr–Purcell sequence appears to remove all noticeable frequency shift effects.

**COMPARISON TO EXPERIMENT**

The simulations presented in Figs. 1–7 demonstrate both the accuracy and ability of the numerical approach to describe NMR transients reflecting a nucleus having both a single quadrupolar coupling and a distribution of quadrupolar couplings moving in an inhomogeneous static magnetic field. Most interesting molecular structures, however, have several nuclei with different frequency offsets or chemical shifts and different site specific residual couplings in addition to inhomogeneous and homogeneous broadening reflecting both spin–spin $T_2$ and spin–lattice $T_1$ relaxation. To account for these shifts, splittings, and signal decays, a separate calculation must be done for each chemically inequivalent site. The transients for each site are then appropriately weighted and summed to provide the NMR signal for the whole molecule. The effect of a frequency offset or chemical shift can be included by replacing $\phi(t)$ in the operator $R_z(t)$ shown in the Appendix with $\phi(t) + \sigma t$ where $\sigma$ is the isotropic chemical shift. Relaxation is included in a phenomenological fashion. The rate $1/T_2$ is used to describe damping of the off-diagonal elements of the density operator $\rho[t]$ while the rate $1/T_1$ accounts for the decay of the diagonal components. Therefore, the state vectors $|I_z\rangle$, $|I_y\rangle$, $|a_+\rangle$, $|a_-\rangle$, $|dqr\rangle$, and $|dq\rangle$ are taken to decay at the rate $1/T_2$, $|I_z\rangle$ relaxes at $1/T_1$, and $|I_y\rangle$ repopulates to the initial value of 1 as $-(|I_z\rangle - 1)/T_1$.

The one-dimensional NMR spectrum of $^{14}$C$_6$D$_3$NO$_2$ contains three peaks of corresponding to the ortho, meta, and para deuterons, respectively. The intensities of these peaks are in a 2:2:1 respective ratio, and the residual quadrupolar coupling depends upon the square of the applied electric field. For the para site, the residual quadrupolar coupling as a function of field is $Q_p = (4.8 \times 10^{-3} \text{ cm}^2\text{Hz/kV}^2)E^2$ while $Q_o = Q_m = (1.8 \times 10^{-3} \text{ cm}^2\text{Hz/kV}^2)E^2$ for the ortho and meta sites. Typically in our experiments, the spectrometer operating frequency is chosen such that the electric field independent isotropic chemical shifts of the ortho, meta, and para deuterons are $\sigma_o = +15 \text{ Hz}$, $\sigma_m = -15 \text{ Hz}$, and $\sigma_p = +5 \text{ Hz}$, respectively. Therefore, in zero-applied electric field there are only three peaks observed that are centered at these particular frequencies as shown in Fig. 8(A). As the electric field is increased over the range of 0 to 44 kV/cm these three peaks narrow and split as shown in Fig. 8, reflecting molecular motion and the values of $Q_o$, $Q_m$, and $Q_p$. The presence of motion is confirmed by examining the amplitude of the spin-echo signal as a function of the applied electric field. In zero-applied electric field the echo decay is exponential with rate $4.76 \text{ s}^{-1}$ indicating that $T_2 = 210 \text{ ms}$ for each deuterium site. For higher fields such as $E = 28 \text{ kV/cm}$, this decay rate increases to $19.5 \text{ s}^{-1}$, confirming the presence of motion. Under no conditions could beats on the spin-echo signal like those predicted in Fig. 2 be observed. Presumably this is because the envelope decay due to motion induced by $E$ is always faster than any frequencies recovered by $E$. It is for this reason that the Carr–Purcell experiment was used to generate transients like those shown in Fig. 9. In the absence of motion in $^{14}$C$_6$D$_3$NO$_2$ and for perfectly homogeneous rf pulses, the dashed line shown in Fig. 9(A) illustrates the anticipated result with $T_2 = 210 \text{ ms}$ for a Carr–Purcell echo train having $72 \pi$ pulses separated by 6 ms with single-point detection at each echo maximum. As expected, the echo intensity modulates at the three residual quadrupolar coupling frequencies and all effects of frequency offset have been refocused. The solid line shown in Fig. 9(A) is an example of experimental data obtained for $^{14}$C$_6$D$_3$NO$_2$ at $E = 27.6 \text{ kV/cm}$ using the same 6 ms delay $72 \pi$ pulse Carr–Purcell experiment. Clearly there is no sign of echo modulation due to residual quadrupolar couplings in the ortho, meta, and para sites. Indeed, regardless of field (up to $E = 80 \text{ kV/cm}$ where $Q_p = 60 \text{ Hz}$), the Carr–Purcell echo train never displayed beats suggesting residual quadrupolar couplings, even though one-dimensional spectra indicate these additional splittings. Instead, the Carr–Purcell echo train signal decay rate increased largely with the square of the applied electric field. Evidence of this dependence is provided in Fig. 9(B) where
Figure 8  Examples of some experimental one-dimensional $^2$H NMR spectra of C$_6$D$_5$NO obtained in this study for $E = 0, 6, 11$ kV/cm, and $44$ kV/cm in (A), (B), (C), and (D), respectively. Motional narrowing of the peaks is observed in (B), (C), and (D) while residual quadrupolar splittings clearly manifest themselves in (C) and (D).

choosing $\omega_a = 55$ rad/s and by using inhomogeneous rf pulses, here Gaussian with half field at the edge of the sample. The upper abscissa in Fig. 9(B) shows the exchange rate $\Gamma$ used for the Gaussian conditional probability distribution.

The presence of rf inhomogeneity is suggested by the disagreement between the $E = 0$ kV/cm decay rate of $4.1$ s$^{-1}$ shown in Fig. 9(B) and the $E = 0$ kV/cm decay rate of $4.76$ s$^{-1}$ observed with the single spin echo experiment. The effects of inhomogeneous quadrupolar couplings are most likely not producing the observed decay rate because the free induction signal always displays longer lifetimes as a function of increased electric field, an observation contrary to the simulations shown in Fig. 6. Finally, to remain consistent with
the experimental data, the exchange rate $\Gamma$ was chosen to depend on $E$ as $\Gamma = (0.18 \, \text{cm}^2/\text{kV}^2\text{s}) E^2$. This theoretical number can be compared to the measured electric field dependent diffusion coefficient for $\text{C}_6\text{D}_3\text{NO}_2$, neglecting residual Brownian motion $D \approx (3.33 \times 10^{-3} \, \text{cm}^2/\text{kV}^2\text{s}) E^2$ (9). The electric field dependence cancels from the ratio $D/\Gamma$ yielding a length $l = (D/\Gamma)^{1/2} = 1.3 \, \text{mm}$ which is nearly half the separation of the electrode plates. Barring errors in the measurement of $D$ from imaging data (9) and errors in the simulation using $\Gamma$ in Fig. 9(B), one might expect that the $(D/\Gamma)^{1/2}$ ratio would actually be the sample length, not half the distance as determined here. One explanation for this discrepancy is that there are indeed errors in the measurement of $D$ and determination of $\Gamma$ from simulation. Another likely explanation is that the measurement of $D$ is based on regressing image profiles to a proper diffusion probability with reflecting boundaries, while determination of $\Gamma$ is based on a Gaussian probability only. Here, as mentioned above, exchange is permitted within the Gaussian line shape, but reflecting boundaries are not included. A better measure of $\Gamma$ would involve a simulation yielding a Gaussian spectrum at all times yet also having reflecting boundaries. Determination of $\Gamma$ in this way would most likely shift the $(D/\Gamma)^{1/2}$ ratio to the expected cell length.

Having identified rf inhomogeneity as the source of this motional smearing of residual couplings, it is useful to speculate on how to compensate for this effect in NMR measurements. Unfortunately composite $\pi$ rf pulses and inhomogeneity-compensated rf pulses will not work because the time scale and bandwidth selective nature is too long and narrow when the increased motion due to the electric field is admitted. A solution implemented in our laboratory is to operate the Carr–Purcell experiment in the two-dimensional mode. Here the time gained from each additional $\pi$ rf pulse forms a second dimension and two $\pi/2$ rf pulses applied at the echo maximum and phase cycled with the receiver to select double quantum coherences are used. The first or direct dimension is constructed from the full echo signal from the maximum of the spin echo until the signal has decayed to zero. Operation in this way selects only the fraction of the signal evolving under residual quadrupolar couplings in the indirect dimension and compensates for the effects of motion and inhomogeneous rf on the signal in the indirect dimension.

**SUMMARY**

The purpose of this paper was to introduce a numerical extension of an analytical formalism for handling the effects of diffusion on NMR transient signals and use the approach to simulate experimental results on electrically aligned liquids. To achieve these goals, the numerical approach was compared with standard analytical results obtained using an exponential correlation function. The strengths of the numerical technique were demonstrated by examining the consequences of rf inhomogeneity and coupling heterogeneity on transient NMR responses. As a final test, the numerical approach was used to demonstrate that rf inhomogeneity is most likely the cause of the smearing of residual quadrupolar couplings obtained from multipulse transients in electrically aligned liquids.

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**APPENDIX**

The individual rotation operators that comprise $R_m(t)$ in terms of the gyromagnetic ratio $\gamma$, a linear field gradient $\vec{G}$, and position $z(t)$ are computed below in the basis spanned by the nine states in Eq. [1]. The rotation matrix for $R_z(t)$ is produced by transforming the spin Hamiltonian $\hat{H}_z = \gamma G z(t) I_z = \omega(t) I_z$ in the $3 \times 3$ Hilbert space for an $I = 1$ nucleus into the $9 \times 9$ dimensional Liouville space determined by the basis vectors in Eq. [1]. This process is accomplished using the definition $\langle \hat{H}_z(t) | \rho(t) \rangle = -i [\hat{H}_z(t), \rho(t)]$. Since $\hat{H}_z(t)$ commutes with itself at all times, the solution to the Liouville equation in Liouville space
defines $R_z(t)$ as

$$R_z(t) = \exp\left(-i \int_0^t \mathcal{H}_z(t) \, dt \right)$$

\[
\begin{bmatrix}
|I_x\rangle & |I_y\rangle & |I_z\rangle & |a_x\rangle & |a_y\rangle & |I_x^2\rangle & |dqr\rangle & |dqi\rangle & |I^2\rangle \\
\cos \phi(t) & \sin \phi(t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sin \phi(t) & \cos \phi(t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \phi(t) & \sin \phi(t) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sin \phi(t) & \cos \phi(t) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos 2\phi(t) & \sin 2\phi(t) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin 2\phi(t) & \cos 2\phi(t) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

where $\phi(t) = \int_0^t \omega(t) \, dt$. Similar reasoning can be applied to both the residual quadrupolar coupling Hamiltonian $\mathcal{H}_Q = \hat{Q}(3I_x^2 - I^2)$ where $\hat{Q}$ is the average quadrupolar coupling constant and the rf pulse Hamiltonian $\mathcal{H}_\gamma = d\theta/dt$ in the $3 \times 3$ Hilbert space. The resulting rotation matrices are

$$R_Q(t) = \exp\left(-i \mathcal{H}_Q \right)$$

\[
\begin{bmatrix}
|I_x\rangle & |I_y\rangle & |I_z\rangle & |a_x\rangle & |a_y\rangle & |I_x^2\rangle & |dqr\rangle & |dqi\rangle & |I^2\rangle \\
\cos \pi Qt & 0 & 0 & 0 & 0 & \sin \pi Qt & 0 & 0 & 0 \\
0 & \cos \pi Qt & 0 & -\sin \pi Qt & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sin \pi Qt & 0 & \cos \pi Qt & 0 & 0 & 0 & 0 & 0 \\
-\sin \pi Qt & 0 & 0 & 0 & 0 & \cos \pi Qt & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and

$$R_\gamma(t) = \exp\left(-i \mathcal{H}_\gamma \right)$$

\[
\begin{bmatrix}
|I_x\rangle & |I_y\rangle & |I_z\rangle & |a_x\rangle & |a_y\rangle & |I_x^2\rangle & |dqr\rangle & |dqi\rangle & |I^2\rangle \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & 0 & 0 & 0 & 0 & -\sin \theta \\
0 & 0 & 0 & 0 & \cos 2\theta & 3 \sin 2\theta/2 & \sin 2\theta/2 & 0 & -\sin 2\theta/2 \\
0 & 0 & 0 & 0 & -\sin 2\theta/2 & \cos^2\theta - \sin^2\theta/2 & -\sin^2\theta/2 & 0 & \sin^2\theta/2 \\
0 & 0 & 0 & 0 & -\sin 2\theta/2 & -3 \sin^2\theta/2 & \cos^2\theta + \sin^2\theta/2 & 0 & \sin^2\theta/2 \\
0 & 0 & 0 & \sin \theta & 0 & 0 & 0 & \cos \theta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
where it is understood that $\theta$ is the instantaneous value of $\theta$ at the time $t$. The basic building block for the spin-echo and Carr–Purcell experiments is therefore the product $R_{0}(\tau)R_{q}(\tau)R_{0}(\tau)R_{q}(\tau)$. Note that when $z(t)$ is time independent, i.e., $z(t) = z$ or $G = 0$ and $\theta = \pi$, $R_{n}$ is diagonal with elements 1. For $Q = 0$ this indicates that the signal is identical to the initial signal at the time of the echo. For $Q \neq 0$, the echo signal $\langle I_{p}(t) \rangle$ modulates at $Q$.

REFERENCES

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